Rationality of forms of $M_{0,n}$

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Geometric background

The moduli spaces

$M_{0,n}$ moduli space of n-pointed curves of genus 0, i.e., $(\mathbb{P}^1, p_1, \ldots, p_n)$ where the $p_i \in \mathbb{P}^1$ are distinct, up to projective equivalence

$\overline{M}_{0,n} \supset M_{0,n}$ compactification by stable curves $(C, p_1, \ldots, p_n)$ where C is a nodal tree of $\mathbb{P}^1$'s and the p_i are distinct smooth points. Stability means that each irreducible component has ≥ 3 distinguished points i.e. $\omega_C(p_1 + \cdots + p_n)$ is ample

Group actions

$\mathfrak{S}_n$ symmetric group on $\{1, 2, \ldots, n\}$

This acts on the moduli spaces

$$\begin{array}{rcl} \mathfrak{S}_n \times M_{0,n} & \longrightarrow & M_{0,n} \\ \sigma \cdot (\mathbb{P}^1, p_1, \ldots, p_n) & \mapsto & (\mathbb{P}^1, p_{\sigma(1)}, \ldots, p_{\sigma(n)}) \end{array}$$

giving

$$\mathfrak{S}_n \hookrightarrow \mathrm{Aut}(\overline{M}_{0,n}).$$

Bruno and Mella prove equality for $n \geq 5$; Royden has similar results for $M_{0,n}$.

Rationality

$M_{0,n}$ is rational but the constructions break symmetry:

Keel construction: There is a unique $\phi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ with

$$\phi(p_1) = [1,0],\ \phi(p_2) = [0,1],\ \phi(p_3) = [1,1].$$

This induces a birational morphism

$$\begin{array}{rcl} \beta_{123} : M_{0,n} & \longrightarrow & (\mathbb{P}^1)^{n-3} \\ (\mathbb{P}^1, p_1, \ldots, p_n) & \mapsto & (\phi(p_4), \ldots, \phi(p_n)) \end{array}$$

extending naturally to $\overline{M}_{0,n}$. For instance,

$$\overline{M}_{0,4} \simeq \mathbb{P}^1, \quad \overline{M}_{0,5} \simeq \mathrm{Bl}_{\text{three points}}(\mathbb{P}^1 \times \mathbb{P}^1).$$

Losev-Manin construction: Choosing any ϕ with $\phi(p_1) = [1,0]$ and $\phi(p_2) = [0,1]$ induces

$$\begin{array}{rcl} \beta'_{12} : M_{0,n} & \longrightarrow & (\mathbb{P}^1)^{n-2}/\mathbb{G}_m \\ (\mathbb{P}^1, p_1, \dots, p_n) & \mapsto & (\phi(p_3), \dots, \phi(p_n))/\text{scaling} \end{array}$$

where the multiplicative group acts diagonally on the factors fixing $[1,0]$ and $[0,1]$. This yields toric models of the moduli space.

Kapranov construction: Suppose we have only a single point p_1. Kapranov describes an explicit blowup

$$\beta_1'' : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$$

with center supported in linear subspaces spanned by

$$p_2 = [1, 0, \ldots, 0], \cdots, p_{n-1} = [0, \ldots, 0, 1], p_n = [1, \ldots, 1] \in \mathbb{P}^{n-3}.$$

Indeed, take the linear series

$$|\omega_C(p_2 + \cdots + p_n)| : C \to \mathbb{P}^{n-3}$$

and move the marked points to the prescribed locations. For instance,

$$\overline{M}_{0,4} \simeq \mathbb{P}^1, \quad \overline{M}_{0,5} \simeq \mathrm{Bl}_{\text{four points}}(\mathbb{P}^2).$$

Equivariance

The Keel construction is compatible with actions of

$$\mathfrak{S}_3 \times \mathfrak{S}_{n-3} \subset \mathfrak{S}_n.$$

The Losev-Manin construction is compatible with

$$\mathfrak{S}_2 \times \mathfrak{S}_{n-3} \subset \mathfrak{S}_n.$$

The Kapranov construction with

$$\mathfrak{S}_{n-1} \subset \mathfrak{S}_n.$$

When multiple constructions apply they induce Cremona transformations, e.g., the Kapranov constructions

$$\mathbb{P}^2 \xleftarrow{\beta_1''} \overline{M}_{0,5} \xrightarrow{\beta_2''} \mathbb{P}^2$$

give a birational morphism

$$\overline{M}_{0,5} \to \mathrm{Graph}(\beta_2'' \circ (\beta_1'')^{-1})$$

to a toric variety realizing the Losev-Manin morphism

$$\beta_{12}' : \overline{M}_{0,5} \to (\mathbb{P}^1)^4/\mathbb{G}_m.$$

Gelfand-MacPherson correspondence

One final construction with $\mathfrak{S}_n$ equivariance: Consider $\mathrm{Mat}(2, n)$ the $2 \times n$ matrices. Let GL_2 act from the left and $\mathbb{G}_m^n$ act from the right via diagonal matrices:

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \end{pmatrix} \begin{pmatrix} t_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_n \end{pmatrix}.$$

We have

$$\mathrm{PGL}_2 \backslash (\mathbb{P}^1)^n = \mathrm{GL}_2 \backslash \mathrm{Mat}(2, n) / \mathbb{G}_m^n = \mathrm{Gr}(2, n) / T$$

where $T = \mathbb{G}_m^n / \mathbb{G}$. Depending on how you interpret the quotient operations, one obtains $\overline{M}_{0,n}$ or other models of $M_{0,n}$.

Guiding problems

Galois formulation

Fix k a field and $\rho : \mathrm{Gal}(k) \to \mathfrak{S}_n$ a representation. Realizing $\mathfrak{S}_n \subset \mathrm{Aut}(\overline{M}_{0,n})$ we get a **twisted form** ${}^\rho\overline{M}_{0,n}$ defined over k. This is the moduli space of pairs (C, Z) consisting of a reduced cycle of n points on a genus zero nodal curve, all defined over k, where the Galois group acts on the points via ρ.

Question

When is ${}^\rho\overline{M}_{0,n}$ rational or stably rational over k?

Recall that a variety X is **rational** if there is birational $\mathbb{P}^d \overset{\sim}{\dashrightarrow} X$ over k. It is **stably rational** if $X \times \mathbb{P}^r$ is rational for some r.

An explicit example

Let Z have length four, i.e., is the spectrum of an étale algebra of length four over k. Embed

$$Z \subset \mathbb{P}^2_k$$

as a set in general position using the trace-free elements. Then ${}^{\rho}\overline{M}_{0,4}$ parametrizes the pencil of conic plane curves

$$Z \subset C \subset \mathbb{P}^2.$$

It is isomorphic to $\mathbb{P}^1$ over k.

Another example

${}^{\rho}\overline{M}_{0,5}$ is a quintic del Pezzo surface, all of which are shown to be rational by Enriques and Swinnerton-Dyer (and Skorobogatov).

Proof: The Gelfand-MacPherson correspondence gives

$$ {}^{\rho}\overline{M}_{0,5} = \mathrm{Gr}(2,5)/T_{\rho} $$

where the torus is non-split

$$ 1 \to \mathbb{G}_m \to \rho \text{ permutation torus } \to T_{\rho} \to 1. $$

Pick a suitable three-dimensional subspace $V \subset k^5$. The induced

$$ \mathbb{P}^2 \simeq \mathrm{Gr}(2,V) \dashrightarrow \mathrm{Gr}(2,5)/T_{\rho} $$

is birational! The quickest way to see this is by computing

$$ [T_{\rho} \cdot L] \in \mathrm{H}^*(\mathrm{Gr}(2,5),\mathbb{Z}), \quad L \in Gr(2,5) \text{ generic.} $$

Equivariant formulation

Let $G \subset \mathfrak{S}_n$ be a subgroup and consider $\overline{M}_{0,n}$ as a G-variety. Is this linearizable or stably linearizable?

Recall a G-variety X is **linearizable** if it is equivariantly birational to a linear G-representation or perhaps the projectivization of such a representation. It is **stably linearizable** if it becomes linearizable on taking products with such a representation.

Linearizability is the equivariant analog of rationality over non-closed fields.

Examples

For all $G \subset \mathfrak{S}_4$, $\overline{M}_{0,4}$ is linearizable by the proof sketched above.

However, the $\mathfrak{S}_5$ action on $\overline{M}_{0,5}$ is not linearizable as $\mathfrak{S}_5$ lacks the required representations of small dimension. However, the argument sketched above proves it is stably linearizable.

For simplicity, we focus on the Galois-theoretic results for the rest of this talk.

Prior work

Florence and Reichstein have important results that were the point of departure for our work:

Assume k is infinite and take $X = {}^{\rho}\overline{M}_{0,n}$ a twisted form over k.

- $X(k) \neq \emptyset$ and X is unirational over k;
- for odd n, X is always rational over k;
- for even $n \geq 6$ and suitable k, there are examples of non-rational X.

For the last result, they require that $\mu_4 \subset k$ and $\mathrm{Br}(k)[2] \neq \emptyset$. They consider moduli of pairs (C, Z) where C is a non-split conic associated with the Brauer class.

Our results

Let k be an arbitrary field – finite or infinite – and $\rho : \mathrm{Gal}(k) \to \mathfrak{S}_n$ a representation. Let $X = {}^{\rho}(\overline{M}_{0,n})$ be the associated twisted form.

Theorem

If ρ factors through $\mathfrak{S}_{n-1}$ then X is rational over k, via the Kapranov construction.
If ρ factors through $\mathfrak{S}_{n-3} \times \mathfrak{S}_3$ then X is rational over k, via the Keel construction.
If ρ has an odd orbit then X is stably rational over k.

If ρ factors through $\mathfrak{S}_{n-2} \times \mathfrak{S}_2$ then X is birationally toric over k, for an explicit non-split torus derived from ρ, via the Losev-Manin construction. There is an explicit criterion, in terms of Galois cohomology, for when X is stably rational.

Using the last construction and the obstruction to rationality $\mathrm{H}^1(\mathrm{Gal}(k), \mathrm{Pic}(X))$, we obtain

Theorem

Suppose that k admits a bi-quadratic extension. For each even $n \geq 6$ there exists a form X of $\overline{M}_{0,n}$ that is not rational over k.

The simplest example is when the Galois action is via

$$G = \langle (12)(56), (34)(56) \rangle .$$

This applies in cases where $\mathrm{Br}(k) = 0$ e.g., function fields of complex curves have trivial Brauer group and admit biquadratic extensions.

Building on contributions of Cheltsov, we find

Theorem

A twisted form X of $\overline{M}_{0,6}$ is rational if and only if the action factors through

- *the $\mathfrak{S}_5 \subset \mathfrak{S}_6$ fixing an element;*
- *an index-ten subgroup leaving a partition $\{1,2,3,4,5,6\} = \{p,q,r\} \cup \{a,b,c\}$ invariant;*
- *a Klein four group conjugate to $\langle (34),(12)(56) \rangle$.*

This includes all stably rational examples.

The rationality constructions are most transparent using the Segre threefold model of X, i.e., as a cubic threefold with ten nodes.

Using the Gelfand-MacPherson construction, Schubert calculus (Klyachko's formula for the classes of orbits closures), and some combinatorial analysis, we obtain

Theorem

Let k be an arbitrary field and X a twisted form of $\overline{M}_{0,n}$ over k. If n is odd then X is rational over k.

The main challenge is to produce suitable subspace $V \subset k^n$ such that the induced

$$\mathrm{Gr}(2, V) \dashrightarrow {}^{\rho}\overline{M}_{0,n}$$

has the degree predicted from intersection theory.

Using arguments via restriction of scalars and forgetting maps

$$\overline{M}_{0,2m} \to \overline{M}_{0,m} \times \overline{M}_{0,m}$$

we find

Theorem

Let $n = 2m$ with m odd and assume that ρ factors through

$$A \times \mathfrak{S}_2 \subset A \wr \mathfrak{S}_2 \subset \mathfrak{S}_{2m},$$

for some $A \subset \mathfrak{S}_m$. Suppose that X is a twisted form of $\overline{M}_{0,2m}$ associated with ρ. Then X is rational.

In particular, forms for the cyclic group $C_{2m} \subset \mathfrak{S}_{2m}$ and forms defined over $\mathbb{R}$ are rational.

Questions

Are all twisted forms arising from cyclic extensions rational?

Are there any non-rational stably rational twisted forms?

Does the existence of an odd orbit guarantee rationality?